MA40S PRECALCULUS
UNIT F – CONICS
CLASS NOTES (Completed)

What Are Conic Sections?

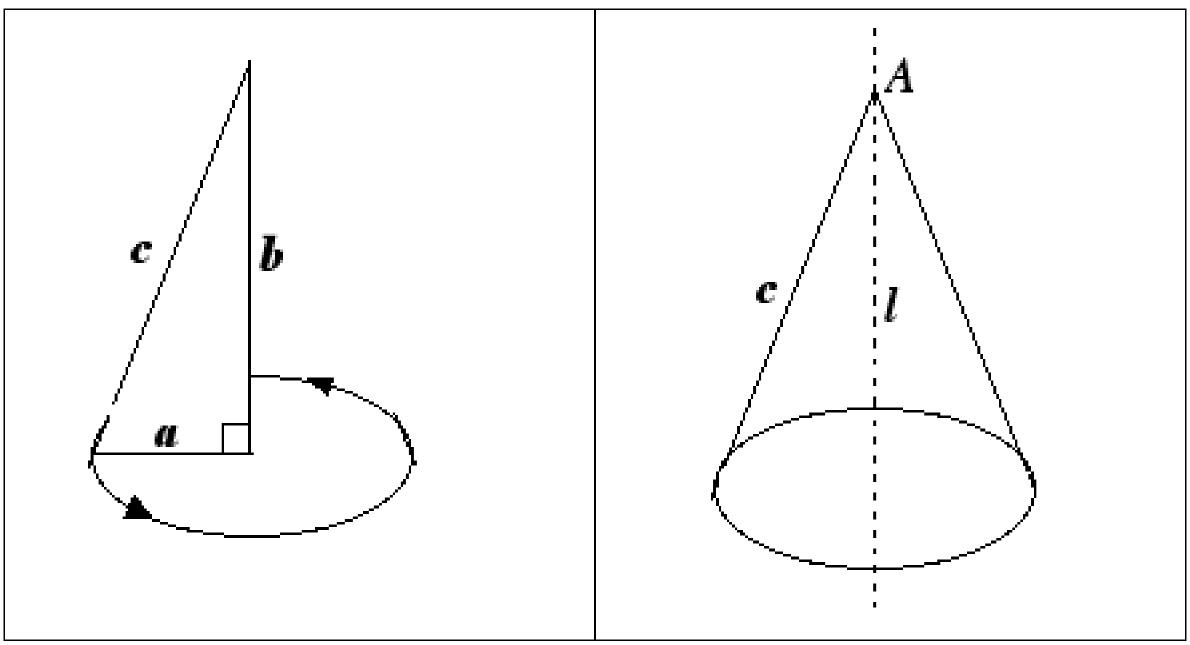
***Objectives*:**

- To visualize the shapes generated from the intersection of a cone and a plane
- For each conic section, to describe the relationship between the plane, the central axis of the cone, and the cone's generator

1) The Cone

Consider a right triangle with hypotenuse *c*, and legs *a,* and *b*.
We can generate a 3–dimensional solid called a *cone* by rotating the triangle about its leg *b*.
The leg *a,* generates a circle, which becomes the *base* of the cone.

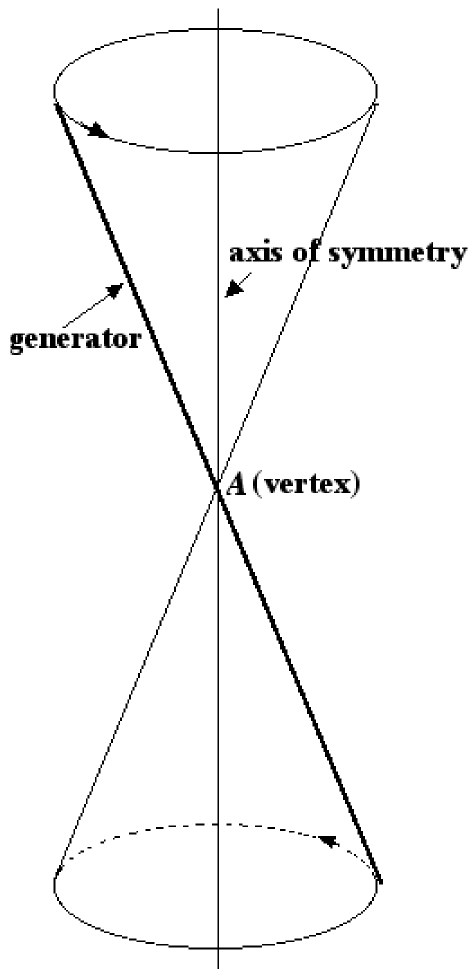
The leg *c* is called the ____***generator***___ of the cone.



The line *l* is the ______***axis of symmetry***____ of the cone.

The point *A* is called the ____***vertex***________ of the cone.

If we place together the vertices of two cones such that they share the same axis of symmetry, we have what is called a ______***double-napped cone***_________.



Label the diagram with the following.

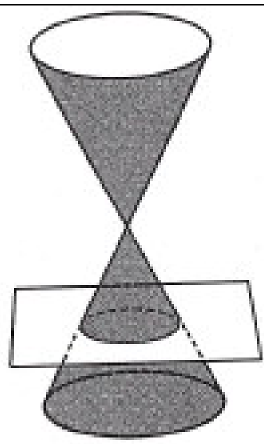
1) vertex *A*

2) axis of symmetry

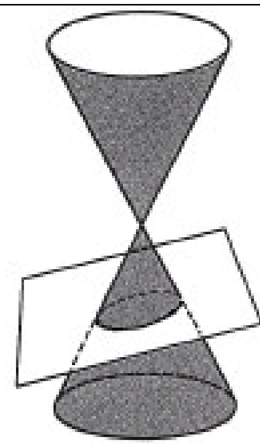
3) generator

2. Conic Sections

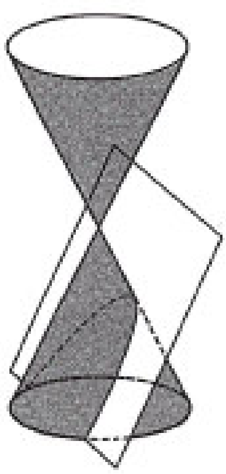
When a plane intersects a cone, it can form two–dimensional objects called conic sections. The type of conic section formed depends upon the angle at which the plane intersects the cone.



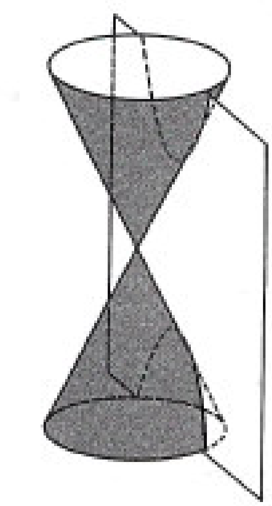
Conic: ***Circle***
The plane is parallel to the base.



Conic: ***Ellipse***
The plane is inclined towards the base of the cone.



Conic: ***Parabola***
The plane is parallel to the generator of the cone.



Conic: ***Hyperbola***
The plane is parallel to the axis of symmetry, and intersects both nappes of the cone. A hyperbola has two distinct branches.

The Circle

***Objectives*:**

- To investigate the standard form and the general form of the equation of a circle
- To determine the radius and centre of a circle given its equation
- To write the equation of a circle given its centre and radius

1. Review

Recall from Principles of Math 11, and from Lesson 1, Chapter 3, the following definition of a circle.

A circle is the set (or locus) of points in a plane that are equidistant from a fixed point.

The fixed point is called the ________***centre***________.
The distance from the centre to any point on the circle is called the ____***radius***________.

Example 1:

Write the equation of the circle with radius 5 and centre (3, 2).

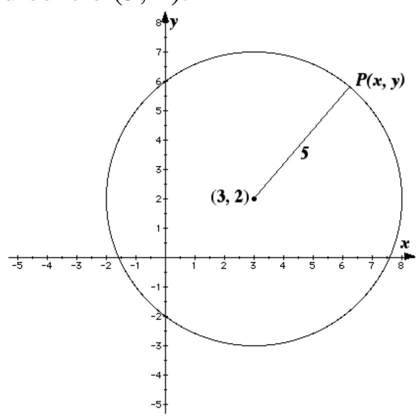
Let $P(x,y)$ be any point on the circle.

Then, using the distance formula, we have

$\sqrt{(x-3)^2+(y-2)^2} = 5^2$.

The equation of the circle is

$(x-3)^2+(y-2)^2 = 25$.



2. The Standard Form of the Equation of a Circle

The following is the standard form of the equation of a circle with radius *r*, and centre (h,k).

$$(x-h)^2+(y-k)^2 = r^2$$

Example 2:

Write the equation of the circle with radius 7 and centre $(-4, 8)$.

We have $h = -4$, $k = 8$, and $r = 7$.

$$(x - {}^{-}4)^2 + (y - 8)^2 = 7^2$$
$$(x + 4)^2 + (y - 8)^2 = 49$$

3. The General Form of the Equation of a Circle

If we expand the equation in Example 2, we obtain the following result.

$$(x + 4)^2 + (y - 8)^2 = 49$$
$$(x^2 + 8x + 16) + (y^2 - 16y + 64) = 49$$
$$x^2 + y^2 + 8x - 16y + 31 = 0$$

For the circle in Example 2, the equation is said to be in the *general form*.

The following is the *general form* of the equation of a circle.

> $Ax^2 + Ay^2 + Dx + Ey + F = 0$
> **where A, D, E, and F are real constants and $A \neq 0$.**

Thus for the circle in Example 2, we have $A = 1$, $D = 8$, $E = -16$, and $F = 31$.

Note: We are reserving the real constants B and C for the general forms of other conics.

Example 3:

Determine the radius and centre of the circle whose general form equation is $x^2 + 10x + y^2 - 6y + 18 = 0$.

To transform $x^2 + 10x + y^2 - 6y + 18 = 0$ into standard form, which tells us the centre and radius of the circle, we use the process called "completing the square."

	Centre: **(–5, 3)** Radius: ***4***

$x^2+10x+y^2-6y+18=0$ $x^2+10x+y^2-6y=-18$ $(x^2+10x+25)+(y^2-6y+9)=-18+25+9$ $(x+5)^2+(y-3)^2=16$	

<u>Example 4:</u>

If m and n are positive integers, determine the radius of the circle $mx^2+my^2-n=0$.

$$mx^2+my^2-n=0$$

$$x^2+y^2=\frac{n}{m}$$

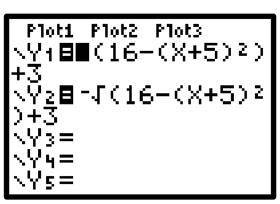
Thus $r=\sqrt{\frac{n}{m}}$

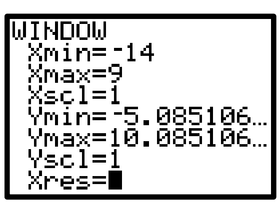
<u>Example 5:</u>

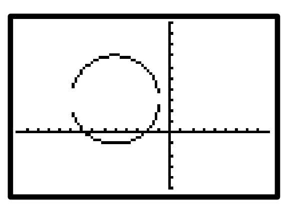
Graph the equation $(x+5)^2+(y-3)^2=16$ using a graphing calculator. (Use the Square viewing window "Zoom Square".) Sketch the result on the grid below.

First, solve the equation for y in terms of x.

$(x+5)^2+(y-3)^2=16$ $(y-3)^2=16-(x+5)^2$ $y-3=\pm\sqrt{16-(x+5)^2}$ $y=\pm\sqrt{16-(x+5)^2}+3$	***This means we must graph two functions:*** $Y_1=+\sqrt{16-(x+5)^2}+3$ **and** $Y_2=-\sqrt{16-(x+5)^2}+3$







The Ellipse

***Objectives*:**

- To investigate the standard form and the general form of the equation of an ellipse
- To graph an ellipse given its equation
- To write the equation of an ellipse given its graph

1. Stretching the Graph of a Circle

Exercise 1:

Let's compare the graph of the circle $x^2 + y^2 = 1$ to the graph of $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$.

Recall the rules for stretching a function $y = f(x)$.

The function $\frac{y}{a} = f(x)$ [or $y = af(x)$] can be obtained by stretching the function $y = f(x)$ according to the following rules.

> If $a > 1$, there is a vertical expansion by a factor of a.
>
> If $0 < a < 1$, there is a vertical compression by a factor of a.

These rules can be applied to relations as well.

What is the effect of replacing the variable y in $x^2 + y^2 = 1$ with $\frac{y}{2}$?

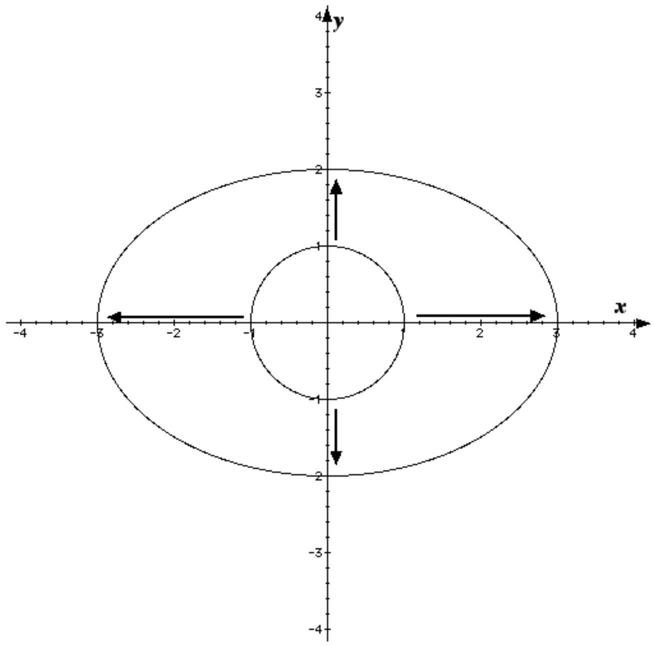
***There is a vertical expansion by a factor of 2*.**______________________________

The same rule will apply to the x–variable.

What is the effect of replacing the variable x in $x^2 + y^2 = 1$ with $\frac{x}{3}$?

***There is a horizontal expansion by a factor of 3*.**______________________________

Given the graph of $x^2 + y^2 = 1$ below, sketch the graph of $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$.



The figure resulting from the two expansions is called an **ellipse**.

List the features of this ellipse.

Domain:

$\boldsymbol{-3 \le x \le 3}$

Range:

$\boldsymbol{-2 \le y \le 2}$

Centre:

(0,0)

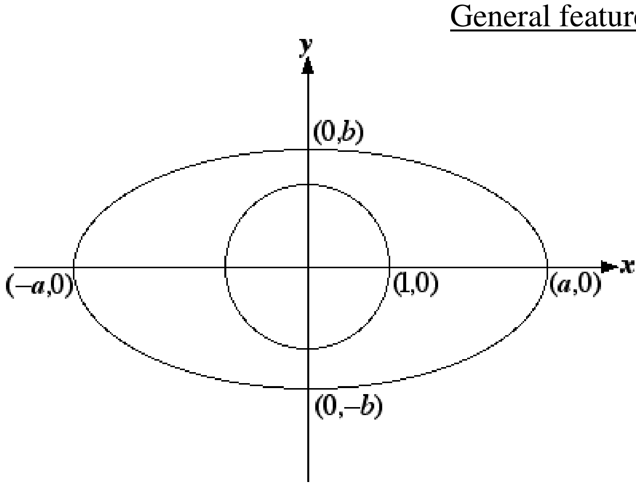
y–intercepts:

2 and –2

x–intercepts:

3 and –3

In general, the graph of $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ is an ellipse, which can be obtained by expanding (or contracting) the circle $x^2 + y^2 = 1$.



General features of the ellipse:

Domain:

$\boldsymbol{-a \le x \le a}$

Range:

$\boldsymbol{-b \le y \le b}$

Centre:

(0,0)

x–intercepts: Let $y = 0$.

$$\boldsymbol{\left(\frac{x}{a}\right)^2 + \left(\frac{0}{b}\right)^2 = 1}$$

$$\boldsymbol{x = \pm a}$$

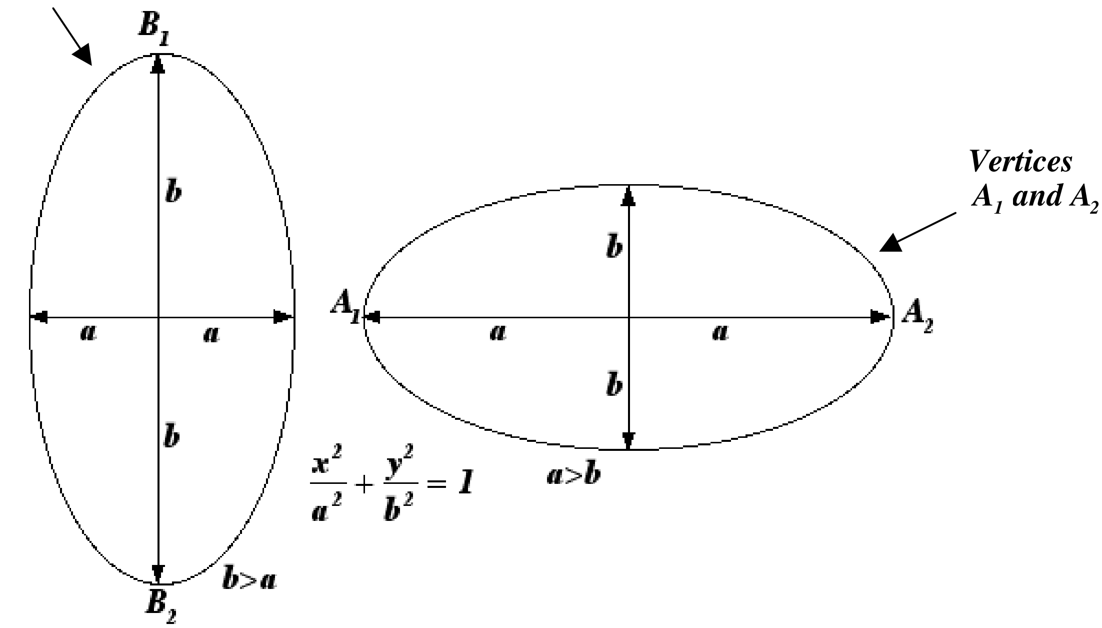
y–intercepts: Let $x = 0$.

$$\boldsymbol{\left(\frac{0}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1}$$

$$\boldsymbol{y = \pm b}$$

Note that the ellipse can be elongated in either the vertical or the horizontal direction.

(Note: Have students label the diagrams as shown below.)
Vertices B_1 and B_2



The longest chord in the ellipse is called the _____***major axis***_________.

The shortest chord in the ellipse is called the _____***minor axis***_________ .

The major axis and the minor axis intersect at the ______***centre***______ of the ellipse.

The endpoints of the major axis are each called a ___***vertex***______ of the ellipse.

Two Cases:

1) When $a > b$, the length of the major axis for $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is ___***2a***_____.

The minor axis will have length ____***2b***_____.

2) When $b > a$, the length of the major axis for $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is ___***2b***_____.

The minor axis will have length ____***2a***_____

Label the vertices and the major and minor axes on both of the above ellipses.

2. Translating the Graph of an Ellipse

Example 2:

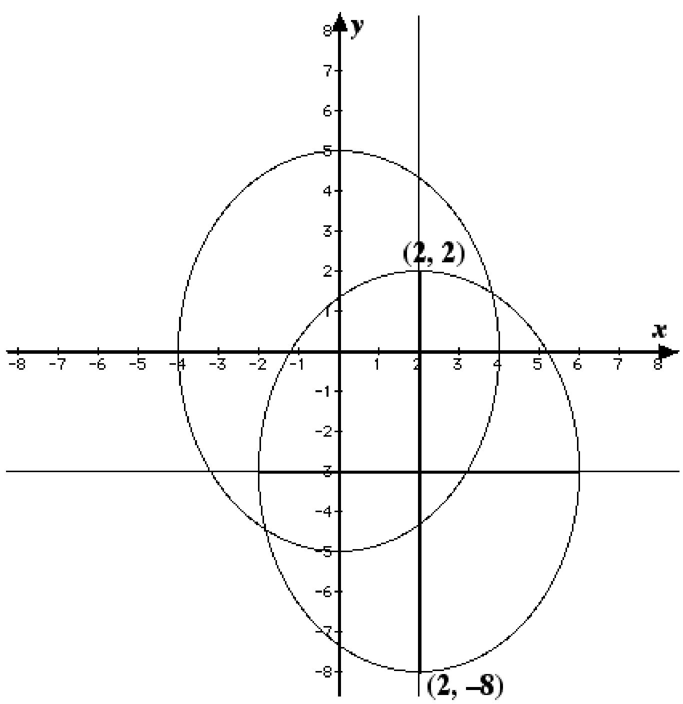
Starting with the graph of $\frac{x^2}{16} + \frac{y^2}{25} = 1$, sketch the graph of $\frac{(x-2)^2}{16} + \frac{(y+3)^2}{25} = 1$.

Identify the translation that occurs when we:

1) Replace the variable x with $(x - 2)$.
________***This means a horizontal translation 2 units to the right.***____________

2) Replace the variable y with $(y + 3)$.
________***This means a vertical translation 3 units down.***_________________

Note that $\frac{x^2}{16} + \frac{y^2}{25} = 1$ can be rewritten as $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$.



$\frac{(x-2)^2}{16} + \frac{(y+3)^2}{25} = 1$

Length of major axis:
$\boldsymbol{2b = 2 \times 5 = 10}$

Length of minor axis:
$\boldsymbol{2a = 2 \times 4 = 8}$

Centre: ***(2, –3)***______

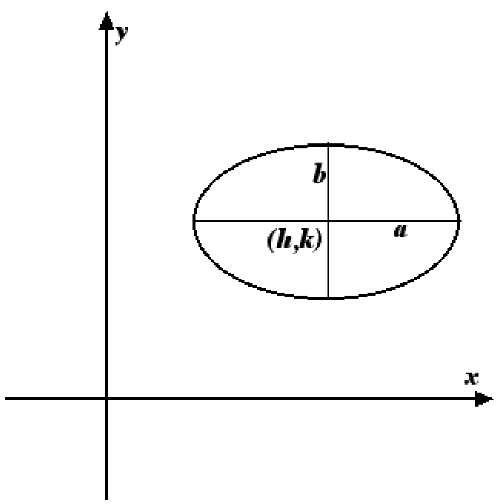
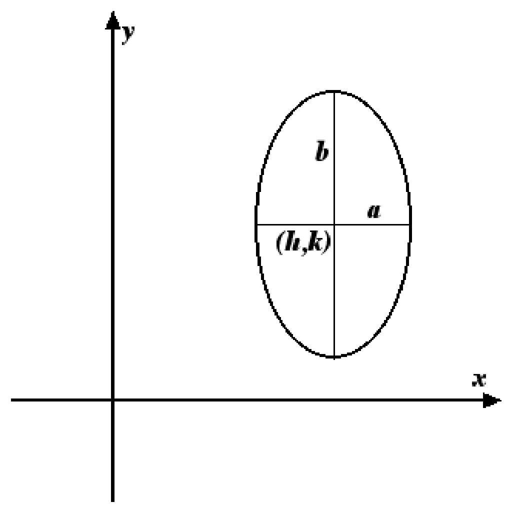
Vertices: ***(2, 2) and (2, –8)***____

3. Standard Form of the Equation of an Ellipse

The following is the *standard form* for an ellipse with centre *(h, k).*

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

There are two cases, depending on whether $a > b$ or $b > a$.

<table>
<tr><th>Ellipse with Horizontal Major Axis ($a > b$)</th><th>Ellipse with Vertical Major Axis ($b > a$)</th></tr>
<tr><td>
</td><td>
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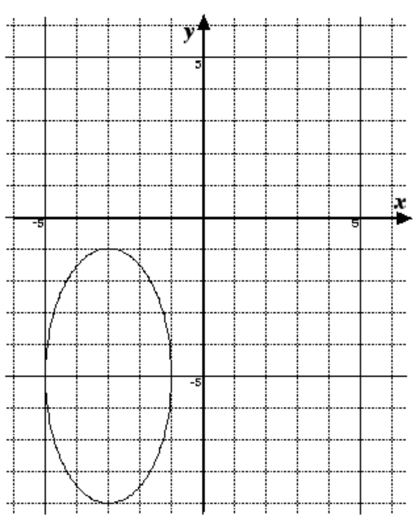
<u>Example 3:</u>

Determine the standard form of the conic graphed below. ***(Exam Specs. #6, p. 26)***



Centre: ***(–3, –5)***

Length of Major axis: ***8***

Value of b: $\boldsymbol{8/2=4}$

Length of Minor Axis: ***4***

Value of a: $\boldsymbol{4/2=2}$

Equation in Standard Form:

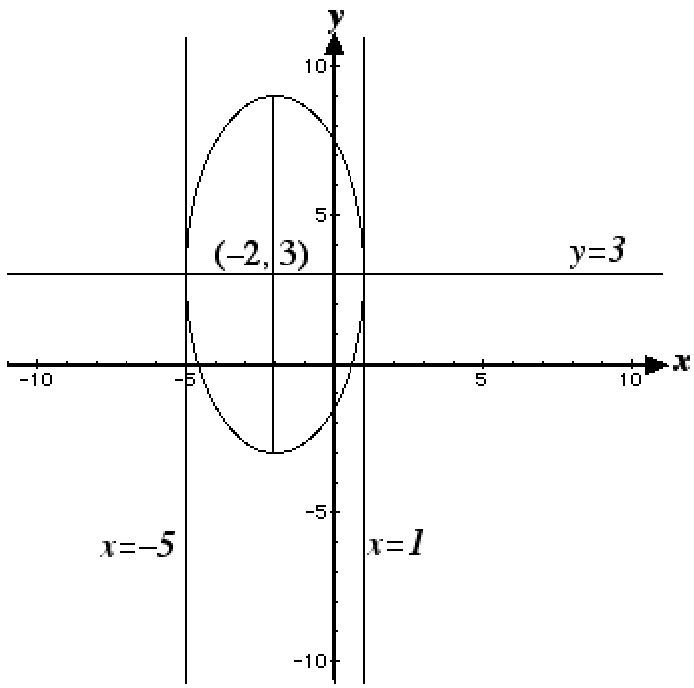
$$\frac{(x-{}^-3)^2}{2^2}+\frac{(y-{}^-5)^2}{4^2}=1 \text{ or } \frac{(x+3)^2}{4}+\frac{(y+5)^2}{16}=1$$

(Sample Exam 2002, #26)

Example 4:

An ellipse is tangent to the lines $x = -5$ and $x = 1$. If the centre of the ellipse is on the line $y = 3$ and the length of the major axis is 12, determine the equation of the ellipse.

On the diagram below, sketch the lines $x = -5$, $x = 1$, and $y = 3$.



Since the ellipse is tangent to the lines $x = -5$ and $x = 1$, the length of the horizontal axis must be $|-5-1| = 6$. This means $2a = 6$ and $a = 3$.

Thus, the vertical axis is the major axis of length 12. This means $2b = 12$ and $a = 6$.

By symmetry, the centre of the ellipse must be located midway between the lines $x = -5$ and $x = 1$. Therefore, its x–coordinate is –2.

Since the centre lies on the line $y = 3$, its coordinates are $(-2, 3)$.

Since the standard form of the equation of an ellipse is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, this ellipse has the equation $\frac{(x+2)^2}{3^2} + \frac{(y-3)^2}{6^2} = 1$ or $\frac{(x+2)^2}{9} + \frac{(y-3)^2}{36} = 1$.

4. General Form of the Equation of an Ellipse

To write the equation $\frac{(x+2)^2}{9}+\frac{(y-3)^2}{36}=1$ in general form, we clear the equation of fractions, and expand the binomial squares.

$$\frac{(x+2)^2}{9}+\frac{(y-3)^2}{36}=1$$

$$36\cdot\frac{(x+2)^2}{9}+36\cdot\frac{(y-3)^2}{36}=36$$

$$4(x+2)^2+(y-3)^2=36$$

$$4(x^2+4x+4)+(y^2-6y+9)=36$$

$$4x^2+16x+16+y^2-6y+9=36$$

$$4x^2+y^2+16x-6y-27=0$$

Note that this result resembles closely the equation for a circle. What is the main difference?

The coefficients of x^2 and y^2 are different.

The general form of the equation of an ellipse has the following form.

$$Ax^2+Cy^2+Dx+Ey+F=0$$

where A, C, D, E, and F are real constants

with $A \neq C$

and $A \cdot C > 0$ (i.e., A and C have the same sign.)

What conic do we have if $A = C$? ***a circle***

Note: We are reserving the real constant *B* for another conic.

The Hyperbola

***Objectives*:**

- To investigate the standard form and the general form of the equation of a hyperbola
- To graph a hyperbola given its equation
- To write the equation of a hyperbola given its graph

1. Investigation

What happens to the graph of the circle $x^2 + y^2 = 16$ when the plus sign is replaced with a minus sign?
In other words, let's investigate the graph of the relation $x^2 - y^2 = 16$.

a) Domain

First, solve the relation $x^2 - y^2 = 16$ for the variable *y*.

$$y^2 = x^2 - 16$$

$$y = \pm\sqrt{x^2 - 16}$$

Thus, we must have $x^2 - 16 \geq 0$ ***and*** $x^2 \geq 16$.

This means the domain is $x \geq 4$ ***and*** $x \leq -4$.

There is a sort of "no man's land" between *x* = 4 and *x* = –4.

b) *y*–intercepts

Substitute $x = 0$ into $y = \pm\sqrt{x^2 - 16}$.

Conclusion:

There are no y–intercepts. This is consistent with the domain restrictions.

c) Range

Solve the relation for the variable *x*.

$$x^2 = 16 + y^2$$

$$x = \pm\sqrt{16 + y^2}$$

Thus, there are no restrictions on the values of *y*.

d) *x*–intercepts

Substitute $y = 0$ into $x = \pm\sqrt{16 + y^2}$.

Conclusion:

***The x*–intercepts are** ± 4.

e) Symmetry of $y = \pm\sqrt{x^2 - 16}$

Symmetry about the *y*–axis:

Both *x* and –*x* can produce the same value of *y*.

Symmetry about the *x*–axis:

Each value of *x* will produce 2 values of *y*.

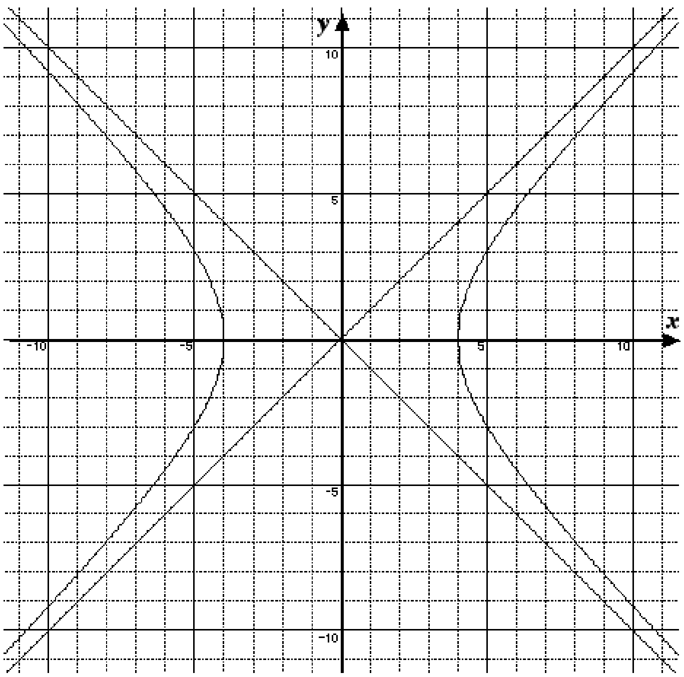
f) Calculating some points

Use the equation $y = \pm\sqrt{x^2 - 16}$ to compute the *y*–values in the table below.

x	4	–4	5	–5	6	–6	7	–7
y	***0***	***0***	**±3**	**±3**	***±4.472***	***±4.472***	***±5.745***	***±5.745***

x	8	–8	9	–9	10	–10
y	***±6.928***	***±6.928***	***±8.062***	***±8.062***	***±9.165***	***±9.165***

Plot these points on the grid below.



The relation $x^2 - y^2 = 16$ is called a *hyperbola*.

Notes:

1) ***A hyperbola has two distinct, unconnected branches.***
2) ***The branches are congruent.***
3) ***There is no curve at all on the interval*** $-4 < x < 4$.

g) Asymptotic Behaviour

Use the equation $y = \pm\sqrt{x^2 - 16}$ to compute the y–values in the table below.

x	100	1000	10000	100000	1000000
y	***99.919968***	***999.9920***	***9999.9992***	***99999.99992***	***1000000***

What happens to the y–values as the x–values get very large?

As the x–values become very large, the values of x^2 become even larger. Thus, the number 16 becomes more and more insignificant when we subtract it from larger and larger values of x^2.

This means that as x gets larger and larger, the value of $\sqrt{x^2 - 16}$ gets closer and closer to $\sqrt{x^2}$ or x itself.

So in quadrant I, the branch $y = \sqrt{x^2 - 16}$ gets closer and closer to the line $y = x$. Thus, the line $y = x$ is an asymptote to the curve $y = \pm\sqrt{x^2 - 16}$.

We can get the curve $y = \sqrt{x^2 - 16}$ as close as we like to the line $y = x$, but it will never touch its asymptote.

By symmetry, the same behaviour occurs in the other three quadrants. The other asymptote to the curve $y = \pm\sqrt{x^2 - 16}$ is the line $y = -x$.

Sketch the asymptotes $y = x$ *and* $y = -x$ on the grid.

2. Standard Form for the Equation of a Hyperbola

The hyperbola $x^2 - y^2 = 16$ can be rewritten in the following form.

$$\frac{x^2}{16} - \frac{y^2}{16} = 1 \text{ or } \frac{x^2}{4^2} - \frac{y^2}{4^2} = 1$$

This equation is now in the *standard form* for the hyperbola.

In general, the following is the standard form for the equation of a hyperbola centred at the origin.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

What is the main difference between this equation, and the standard form for the ellipse?

______________________***the subtraction sign***______________________

A hyperbola like $\frac{x^2}{4^2} - \frac{y^2}{4^2} = 1$ where $a = b$ is called by some a *rectangular* hyperbola.

Example 1:

Consider the hyperbola $16x^2 - 9y^2 = 144$.

a) Convert the equation to standard form.

Divide both sides of the equation by 144.

$$\frac{16x^2}{144} - \frac{9y^2}{144} = \frac{144}{144}$$

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

b) Find the range of the relation.

Solve the equation for x:

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

$$\frac{x^2}{9} = 1 + \frac{y^2}{16}$$

$$x^2 = 9\left(1 + \frac{y^2}{16}\right)$$

$$x = \pm 3\sqrt{1 + \frac{y^2}{16}}$$

This means there is no restriction on the values of y. The range includes all real numbers.

c) Calculate the x–intercepts.

Substitute y = 0 into

$$x = \pm 3\sqrt{1 + \frac{y^2}{16}}$$

$$x = \pm 3\sqrt{1 + \frac{0^2}{16}} = \pm 3$$

d) Find the domain of the relation.

Solve the equation for y.

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$
$$\frac{y^2}{16} = \frac{x^2}{9} - 1$$
$$y^2 = 16\left(\frac{x^2}{9} - 1\right)$$
$$y = \pm 4\sqrt{\frac{x^2}{9} - 1}$$

Thus the domain restriction is

$$\frac{x^2}{9} - 1 \geq 0$$
$$x^2 \geq 9$$
$$x \geq 3 \text{ or } x \leq -3.$$

e) Determine the asymptotes of the relation.

We will consider what happens to the value of *y* as the value of *x* increases without bound.

$$y = \pm 4\sqrt{\frac{x^2}{9} - 1}$$

As x gets larger and larger, the number 1 becomes more and more insignificant when we subtract it from larger and larger values of $\frac{x^2}{9}$.

In other words, as x increases, the value of $y = \pm 4\sqrt{\frac{x^2}{9} - 1}$ ***gets closer and closer to*** $\pm 4\sqrt{\frac{x^2}{9}}$ ***or*** $\pm\frac{4x}{3}$.

Thus, the lines $y = \pm\frac{4}{3}x$ ***are asymptotes to the curve*** $y = \pm 4\sqrt{\frac{x^2}{9} - 1}$.

By symmetry, the same behaviour occurs as the value of x gets more and more negative.

f) Graph the hyperbola $\frac{x^2}{3^2} - \frac{y^2}{4^2} = 1$.

Using the facts that the *x*–intercepts are ± 3, and that the asymptotes to the curve are $y = \pm\frac{4}{3}x$, we can graph the relation quite quickly.

The first step is to sketch the rectangle formed by the lines $x = 3$, $x = -3$, $y = 4$, and $y = -4$.
The second step is to draw and extend the diagonals of the rectangle.
This rectangle is important for two reasons.

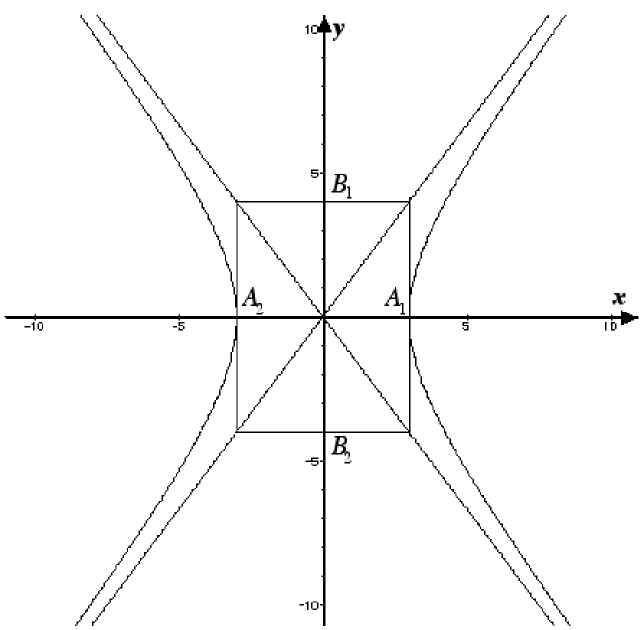
1) ***The graph lies entirely within two of the sectors marked out by the rectangle's diagonals.***

2) ***The diagonals of the rectangle are the asymptotes to the curve.***

Since the hyperbola gets closer and closer to the diagonals, you can use these lines as guides in sketching the curves.

The third step is to plot the *x*–intercepts and another few points for accuracy.

x	4	7	10	–4	–7	–10
y	***± 3.53***	***± 8.43***	***± 12.72***	***± 3.53***	***± 8.43***	***± 12.72***



Label the points $B_1(0,4)$, $B_2(0,-4)$, $A_1(3,0)$ and $A_2(-3,0)$

The points $A_1(3,0)$ and $A_2(-3,0)$ are each called a ***vertex***.

The segment A_1A_2 joining the vertices is the ***transverse axis***.

The segment B_1B_2 is called the ***conjugate axis***.

Example 2:

Graph the relation $\frac{y^2}{3^2} - \frac{x^2}{4^2} = 1$

How is this relation similar to the one in Example 1?

Since the x– and y– variables have been interchanged, the relations are inverses.

Recall from Chapter 1 how the graphs of $\frac{y^2}{3^2} - \frac{x^2}{4^2} = 1$ and $\frac{x^2}{3^2} - \frac{y^2}{4^2} = 1$ are related.

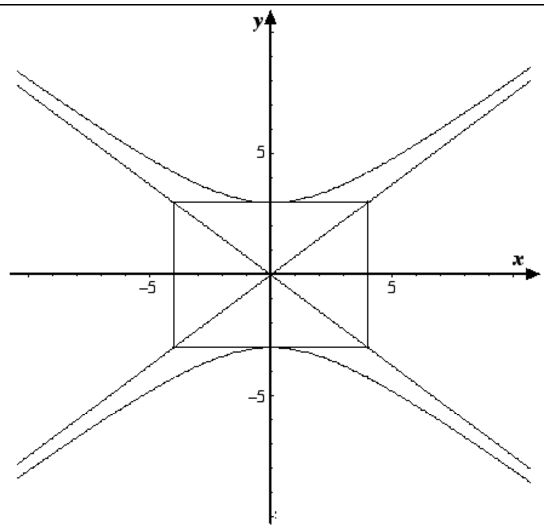
They are reflections of each other in the line $y = x$.

This hyperbola will thus have its branches extending in the *y*–direction.

Step 1
Sketch the rectangle formed by the lines $y = 3$, $y = -3$, $x = 4$, and $x = -4$.

Step 2
Extend the diagonals (asymptotes) of the rectangle.

Step 3
Plot the *y*–intercepts.



What is the length of the conjugate axis?
6

What is the length of the transverse axis?
8

Identify the vertices.
(0, 3) and (0, –3)

(Remember: the transverse axis joins the vertices of the hyperbola.)

3. Translating the Hyperbola

As with the circle and ellipse, the hyperbola can be translated horizontally and vertically.

Example 3:

Graph the hyperbola $\frac{(y+1)^2}{3^2} - \frac{(x-2)^2}{4^2} = 1$.
Identify its vertices and its centre. Find the equations of the asymptotes and the lengths of the axes.

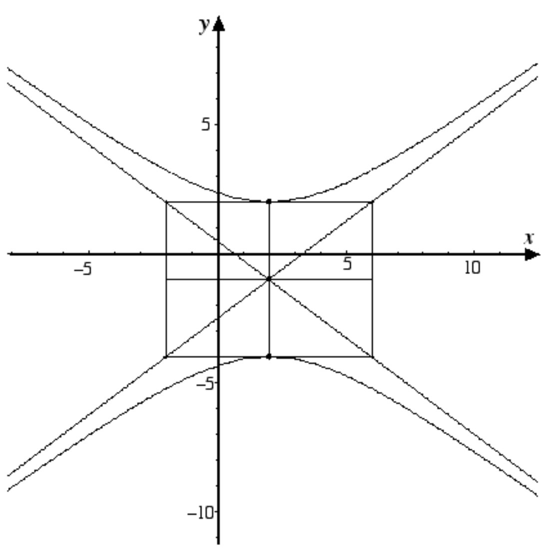
This hyperbola will be congruent to the hyperbola $\frac{y^2}{3^2} - \frac{x^2}{4^2} = 1$. But what is its centre?

Translate the centre (0,0) 2 units right and 1 unit down to get the new centre (2,−1).

What is the length of the conjugate axis? $\underline{\quad 2 \times 4 = 8 \quad}$

What is the length of the transverse axis? $\underline{\quad 2 \times 3 = 6 \quad}$

Our strategy will be to first construct a 6 X 8 rectangle centred at $(2,-1)$.
Then, draw the diagonals (asymptotes).
Finally, draw the axes, plot the vertices, and using the asymptotes as guides, sketch the hyperbola.



Vertices:

(2,2) and (2,−4)

Slopes of the Asymptotes:
Notice from the completed diagram that the slopes are ***3/4 and − 3/4.***

Equations of the Asymptotes:

1) $(y - -1) = -\frac{3}{4}(x-2)$

$y = -\frac{3}{4}x + \frac{1}{2}$

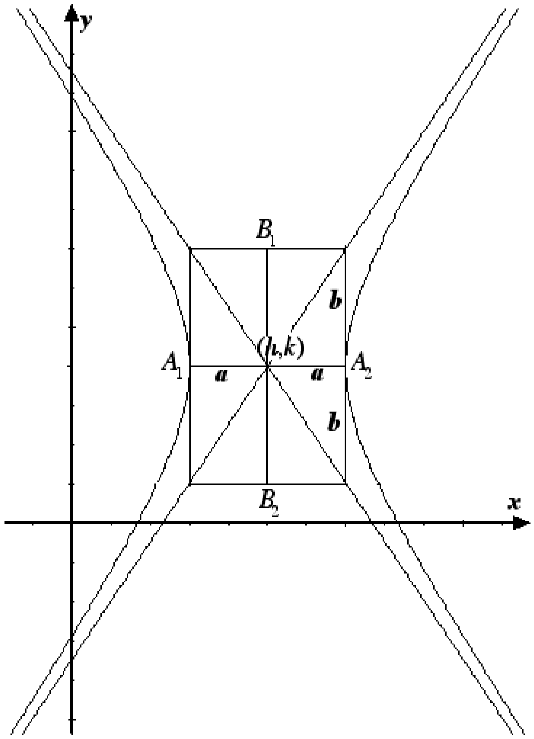
2) $(y - -1) = \frac{3}{4}(x-2)$

$y = \frac{3}{4}x - \frac{5}{2}$

4. Summary of the Standard Form of the Equation of a Hyperbola

The hyperbola can be oriented in two directions.

a) Hyperbola with Horizontal Transverse Axis $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$



Length of Transverse Axis:
$A_1A_2 = 2a$

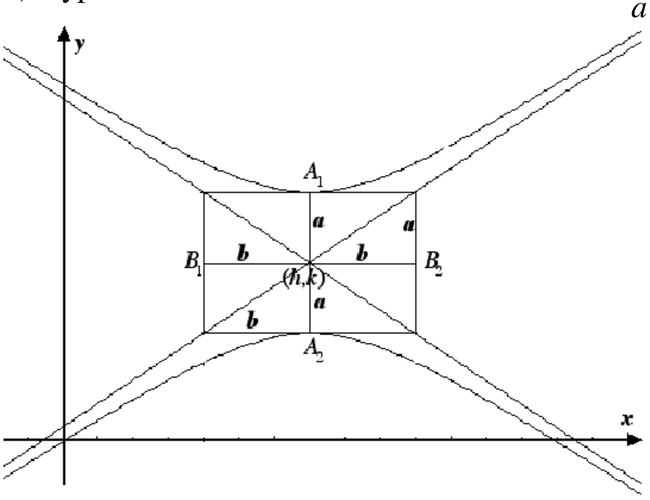
Length of Conjugate Axis:
$B_1B_2 = 2b$

Centre:
(h, k)

Slopes of Asymptotes:

$-b/a$ ***and*** b/a

b) Hyperbola with Vertical Transverse Axis: $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$



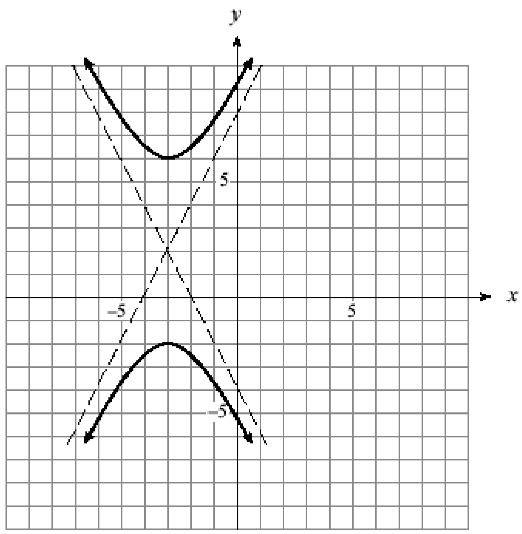
Length of Transverse Axis:
$A_1A_2 = 2a$

Length of Conjugate Axis:
$B_1B_2 = 2b$

Centre: ***(h, k)***

Slopes of Asymptotes:
$-a/b$ ***and*** a/b

Example 4:
Determine the standard form equation of the conic graphed below.



Since the hyperbola is oriented vertically, its standard equation will be of the following form.

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Centre of the hyperbola: $(-3, 2)$

Length of Transverse Axis: 8

Value of a:

$$2a = 8$$
$$a = 4$$

Slopes of Asymptotes:

$$\frac{4}{2} = 2 \text{ and } -\frac{4}{2} = -2$$

Value of b:

Since the slopes of the asymptotes are $\frac{a}{b}$ and $-\frac{a}{b}$ we have $b = 2$.

Equation of the conic:

$$\frac{(y-2)^2}{4^2} - \frac{(x - {}^{-}3)^2}{2^2} = 1$$

$$\text{or } \frac{(y-2)^2}{16} - \frac{(x+3)^2}{4} = 1$$

Note: The above answer may also be written in the form $\frac{(x+3)^2}{4} - \frac{(y-2)^2}{16} = -1$

5. General Form of the Equation of a Hyperbola

The equation $\frac{(y+1)^2}{3^2}-\frac{(x-2)^2}{4^2}=1$ can be expanded and cleared of fractions.

$$\frac{(y+1)^2}{3^2}-\frac{(x-2)^2}{4^2}=1$$
$$144\cdot\frac{(y+1)^2}{9}-144\cdot\frac{(x-2)^2}{16}=144$$
$$16(y+1)^2-9(x-2)^2=144$$
$$16(y^2+2y+1)-9(x^2-4x+4)=144$$
$$16y^2+32y+16-9x^2+36x-36=144$$
$$-9x^2+16y^2+36x+32y-164=0$$

The general form of the equation of a hyperbola has the following form.

$$Ax^2+Cy^2+Dx+Ey+F=0$$

where A, C, D, E, and F are real constants

with $A\cdot C<0$

(i.e., A and C have different signs.)

Remember: We are reserving the real constant *B* for another conic.

Example 5:

Change the following equation to standard form.
$4x^2-y^2+16x+2y-1=0$
Completing the square, we get

$$4(x^2+4x+4)-(y^2-2y+1)=1+16-1$$
$$4(x+2)^2-(y-1)^2=16$$
$$\frac{(x+2)^2}{4}-\frac{(y-1)^2}{16}=1$$

Example 5:

Given the conic $25x^2+16y^2+100x-32y-284=0$,

a) Change the equation to standard form; and
b) Graph the conic on the grid below.

a)

$$25\left(x^2+4x\right)+16\left(y^2-2y\right)=284$$

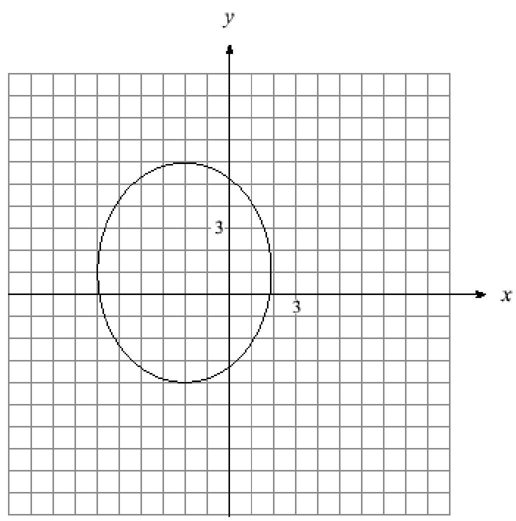
$$25\left(x^2+4x+4\right)+16\left(y^2-2y+1\right)=284+100+16$$

$$25(x+2)^2+16(y-1)^2=400$$

$$\frac{25(x+2)^2}{400}+\frac{16(y-1)^2}{400}=\frac{400}{400}$$

$$\frac{(x+2)^2}{16}+\frac{(y-1)^2}{25}=1$$

b)



The Parabola

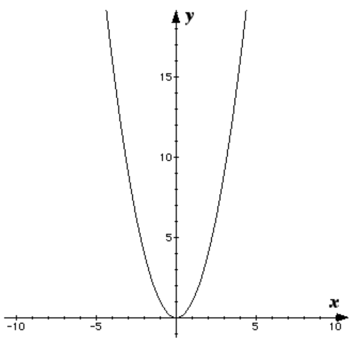
***Objectives*:**

- To investigate the standard form and the general form of the equation of a parabola
- To graph a parabola given its equation
- To write the equation of a parabola given its graph

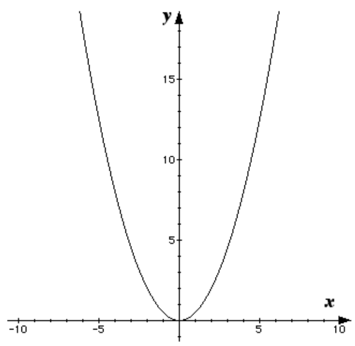
Review

Recall from Chapter 1 the transformations on the simple parabola $y = x^2$.
Graph each function on the grid provided, and indicate the translation(s) you used.

$y = x^2$

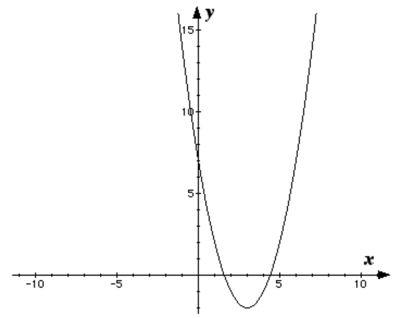


$y = \frac{1}{2}x^2$



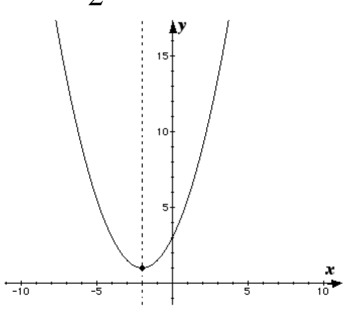
Vertical compression by a factor of 0.5

$y + 2 = (x - 3)^2$



3 units right, 2 units down

$y - 1 = \frac{1}{2}(x + 2)^2$



The Axis of Symmetry:

$\boldsymbol{x = -2}$

Vertex:

$\boldsymbol{(-2, 1)}$

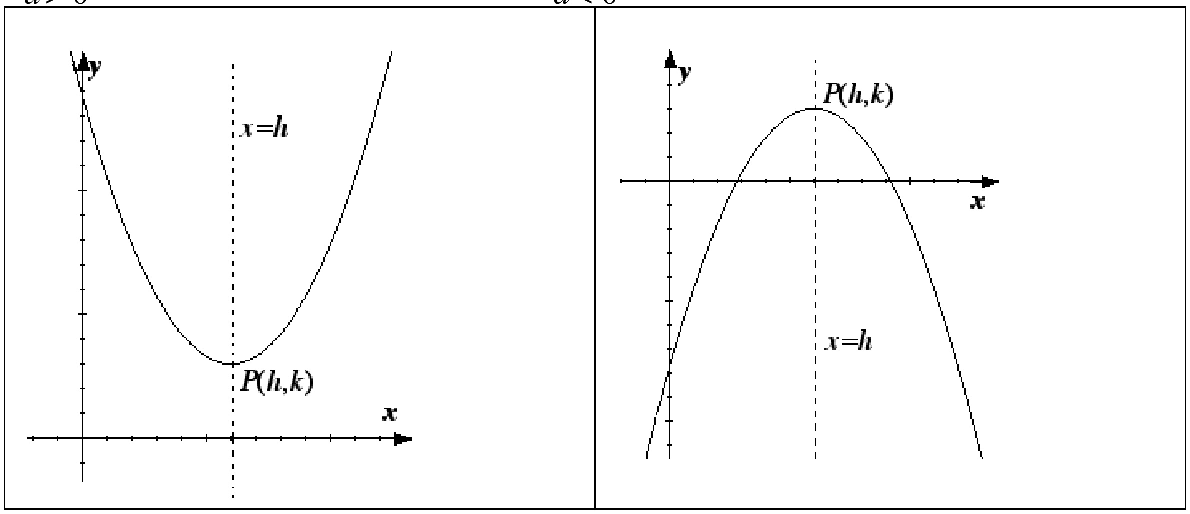
Vertical compression by a factor of 0 . 5, then translate 2 units left, 1 unit up

Standard Form for the Equation of a Parabola

With our knowledge of transformations, we can state the standard form for the equation of a parabola.

<u>Vertical Axis of Symmetry</u> $y-k=a(x-h)^2$ [or $y=a(x-h)^2+k$]

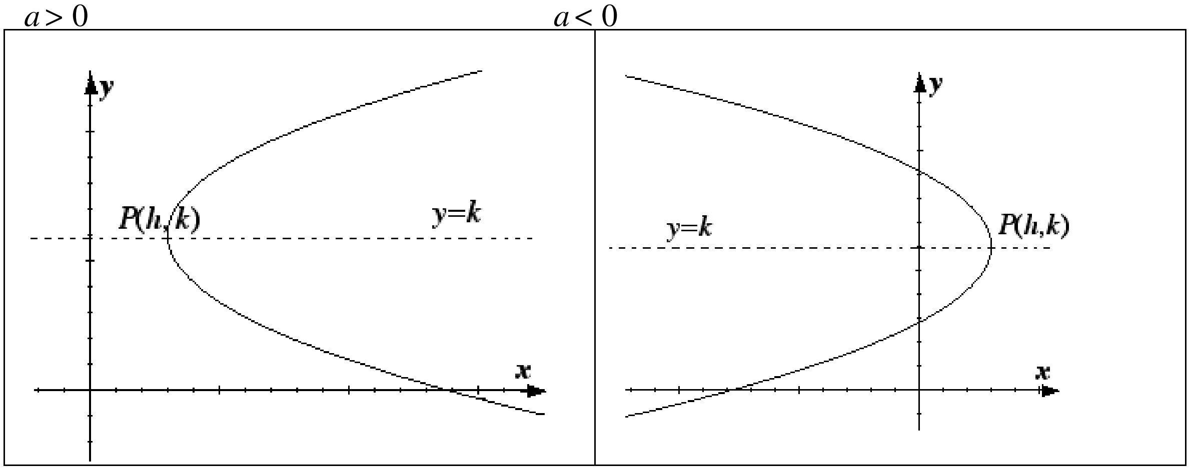
$a>0$ $a<0$



Equation of axis of symmetry: $\boldsymbol{x=h}$ Vertex: ***(h,k)***

<u>Horizontal Axis of Symmetry</u> $x-h=a(y-k)^2$ [or $x=a(y-k)^2+h$]

$a>0$ $a<0$



Equation of axis of symmetry: $\boldsymbol{y=k}$ Vertex: ***(h,k)***

General Form for the Equation of a Parabola

Consider the parabola $y = 2(x-3)^2 - 5$.
Its standard form equation can be expanded and simplified.

$$y = 2(x-3)^2 - 5$$
$$y = 2(x^2 - 6x + 9) - 5$$
$$y = 2x^2 - 12x + 18 - 5$$
$$y - 2x^2 + 12x - 13 = 0$$

This equation is now in *general* form.

The following is the general form for the equation of a parabola.

$Ax^2 + Cy^2 + Dx + Ey + F = 0$ **where**

$A = 0$ **and** $D \neq 0$

or

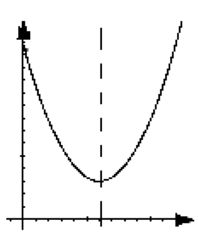
$C = 0$ **and** $E \neq 0$

What happens if $A = 0$ and $D = 0$?
We would lose the x–variable entirely.

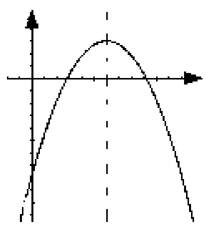
What happens if $C = 0$ and $E = 0$?
We would lose the y–variable entirely.

What is the axis of symmetry's orientation if $C = 0$? ______*vertical*______

This will be a parabola of the family $y = x^2$ *or* $y = -x^2$.



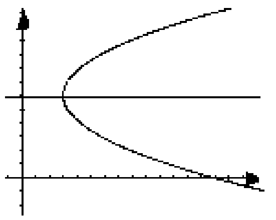
$A > 0$



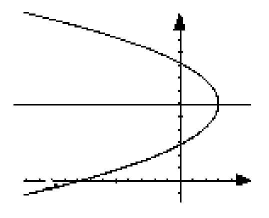
A* < *0

What is the axis of symmetry's orientation if $A = 0$? ______***horizontal***________

This will be a parabola of the family $x = y^2$ ***or*** $x = -y^2$.



C* > *0



C* < *0

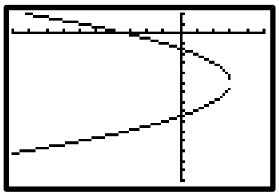
Example 1:

Use a graphing calculator to sketch the curve $y^2 + 10y + 4x + 13 = 0$.

To graph this relation on the calculator, we must solve for y. To solve for y, we need to complete the square.

$$y^2 + 10y + 4x + 13 = 0$$
$$(y^2 + 10y + 25) + 4x + 13 = 25$$
$$(y + 5)^2 = -4x + 12$$
$$(y + 5)^2 = -4(x - 3)$$
$$y + 5 = \pm\sqrt{-4(x - 3)}$$
$$y = \pm\sqrt{-4(x - 3)} - 5$$

Sketch the calculator result.



Vertex: $(3,-5)$

Axis of symmetry: $y=-5$

Standard form of the equation:

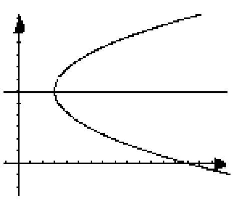
$$(x-3)=-\frac{1}{4}(y+5)^2$$

Example 2:

Give conditions for the constants *A, C,* and *D* such that the following equation is a parabola with a horizontal axis of symmetry. Also, assume that the parabola opens to the right.

$$Ax^2+Cy^2+Dx+y=0$$

This is a parabola of the $x=y^2$ family.



Thus, we must have the following.

$A=0$

$C>0$

$D\neq 0$

What happens if both $A=0$ and $D=0$?

$$Cy^2+y=0$$

$$y(Cy+1)=0$$

Two lines: $y=0$ **and** $y=-\frac{1}{C}$

We call this result a degenerate parabola.